FINITE AMPLITUDE EFFECTS IN RECTANGULAR CAVITIES WITH PERTURBED BOUNDARIES

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THESIS

FINITE AMPLITUDE EFFECTS IN RECTANGULAR CAVITIES WITH PERTURBED BOUNDARIES

by

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The effects of boundary perturbations on finite-amplitude acoustical standing waves in rectangular, rigid-walled cavities were investigated using non-linear theory. When a high amplitude standing wave of frequency ω is generated in a cavity, non-linear effects will cause a stimulation of certain normal modes whose resonance frequencies are integer multiples of ω. Previous experimental observations revealed that there could be excitation of other normal modes, not belonging to the

(20. ABSTRACT Continued)

family of the driven mode, which was not predicted by the non-linear theory.

The purpose of this research was to investigate the possibility that deviations from the idealized geometry could account for these observations. Of the various mechanisms possible, this work investigated the possibility of these unpredicted excitations occurring through a non-linear mechanism.

The standing waves that exist in an ideal cavity must be corrected when the boundaries are irregular. The non-linear interaction between these standing waves and the corrections was studied. The ability of this interaction to excite standing waves other than those predicted in the ideal case was verified. A specific example was worked out demonstrating an unpredicted excitation, the strength of which was on the order of the magnitude of the boundary perturbation parameter.

Finite Amplitude Effects in Rectangular
Cavities with Perturbed Boundaries

by

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LIST OF SYMBOLS

p_0	Classical linear solution for pressure for ideal boundaries
p_1	First order perturbation correction due to boundary irregularities
P_0	Classical linear solution for pressure in a real cavity
$\vec{U}_0$	Classical linear solution for velocity in a real cavity
P_1	First order non-linear perturbation correction for pressure
δ	Magnitude of perturbation parameter on the boundary
L_x, L_y, L_z	Dimensions of the ideal cavity
ρ_0	Density of air
∇	Gradient operator in rectangular coordinates
$\omega=2\pi f$	Angular frequency
c_0	Speed of sound in air
c_p	Phase speed of sound in air
$\Box^2$	D'Lambertian operator
$\Box_L^2$	D'Lambertian operator with losses
γ	Ratio of specific heat at constant pressure to specific heat at constant volume
$(n,m,\ell;q)$	Mode designations
ϕ, θ, β	Phase angles
k_x, k_y, k_z	Propagation constants in the coordinate directions
$k = \sqrt{k_x^2+k_y^2+k_z^2}$	Wave number
M	Peak Mach number of first order ideal solution for p_0
t	Time

RHS	Right hand side of equation
LHS	Left hand side of equation
Q	Quality factor
H	A measure of waveform overlap
L	Loss terms in the non-linear equation

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I. INTRODUCTION

The purpose of this research was to investigate some of the effects of boundary wall perturbations on finite-amplitude standing waves in a rectangular cavity. The investigation was prompted by an examination of the experimental results of Coppens and Sanders [1] and the research of DeVall [2], which suggested the existence of the excitation of modes other than those belonging to the family of the driven mode.

II. BACKGROUND

An acoustical wave traveling in a medium will have its form distorted as predicted by the non-linear hydrodynamic equations. If the medium is absorptive then only waves of relatively high amplitude will develop distortion before becoming completely attenuated due to the absorptive process.

At the United States Naval Postgraduate School, Coppens and Sanders [1], DeVall [2], and others [3,4] have dealt with the study of finite amplitude waves in rigid-walled rectangular cavities and tubes. One interesting result of these cavity experiments was the appearance of excitations of modes which were not family members of the driven mode. For example, assume that the cavity is driven in the (0,1,0;1) mode, which indicates the excitation is of the form $\cos k_y y \cos \omega t$, where

$$k_y = \frac{\pi}{L_y}$$

and

$$k_y^{\,2} = k^2 = (\frac{\omega}{c})^2 \quad .$$

Then a non-familial mode is any which does not have the form (0,n,0;n) . It was proposed [5] that the presence of these non-familial modes could be attributed to the cavity

walls being non-ideal (not perfectly flat) and that perhaps the previous small disparity between theoretical predictions and experimental observations might be partly explained by the presence of these non-familial modes.

The non-linear wave equation applicable to this problem can be expressed [6] as

$$c_0^2 \Box_L^2 \frac{P}{\rho_0 c_0^2} = -\frac{1}{2} \frac{\partial^2}{\partial t^2} \left[\gamma \left(\frac{P}{\rho_0 c_0^2}\right)^2 + \left(\frac{\vec{U}_0}{c_0}\right)^2\right] + \frac{1}{2} c_0^2 \nabla^2 \left[\left(\frac{P}{\rho_0 c_0^2}\right)^2 - \left(\frac{\vec{U}_0}{c_0}\right)^2\right] . \tag{2.1}$$

$\Box_L^2$ is defined as

$$\Box_L^2 = \Box^2 - L ,$$

where L represents the loss terms appropriate to each frequency component in P. The RHS of the non-linear equation may be interpreted as a forcing function (consisting of both classical and non-linearly generated terms) which, in first-order perturbation theory, is the classical solution for the pressure. The first-order perturbation solution may be considered to describe the effects of self-interaction of the classical solution, whereas higher-order perturbation solutions describe the effects of interaction of the

of the non-linear solution with itself. If the cavity is driven at frequency ω, non-linear terms force the existence of all terms containing the form $n\omega$. Each such term whose frequency is close to the resonance frequency of a standing wave of the cavity and whose spatial behavior matches that standing wave can strongly excite that standing wave to a degree determined by Q, the quality factor of the resonance, and by the difference between $n\omega$ and the resonance frequency of the standing wave. In a rectangular cavity, it can be seen [1] that if the classical solution is the $(m_x, m_y, m_z; q)$ mode, then the non-linear wave equation will force the strong excitation of all modes $(nm_x, nm_y, nm_z; nq)$ which constitute the family. Any mode of frequency $n\omega$ which is not a member of the family should not be strongly excited. The purpose of this research is to see if the presence of wall irregularites can explain how non-family members may be strongly excited, and to present an example to support this theory.

III. THE NON-LINEAR PROBLEM

A. CAVITY DESCRIPTION

Assume a perfectly rigid-walled rectangular cavity has one wall perturbed such that the cavity dimensions are $L_x[1+\delta f(y)]$, L_y, L_z . A top view depicting the XY plane and showing the boundary irregularity is given in Fig. 1 below. Also assume the perturbation on the boundary is small compared to the cavity dimensions, $|\delta f(y)| << 1$.

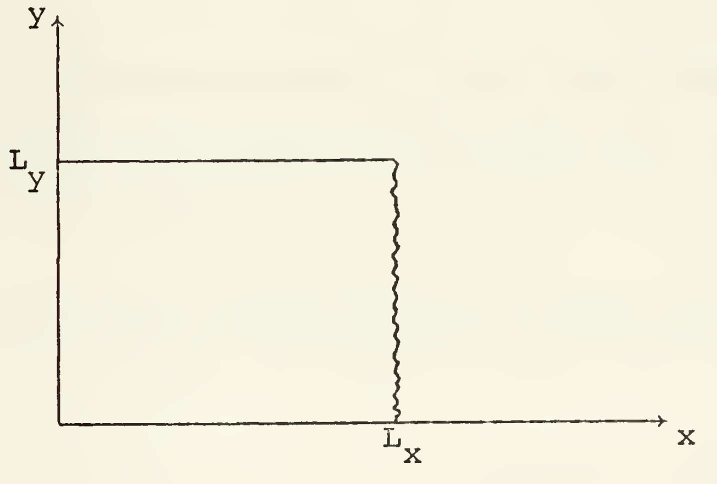

FIGURE 1

The cavity is to be excited by a source near one of the corners in such a way that the $(n,m,\ell;q)$ mode is excited.

B. CLASSICAL SOLUTION FOR IDEAL BOUNDARIES

For a rigid-walled rectangular cavity with ideal boundaries ($\delta=0$), the classical linear solution for the pressure p_0 is subject to the following conditions, namely

$$\Box^2 p_0 = 0 \tag{3.1}$$

and

$$\nabla p_0 \cdot \hat{n}_0 = 0 \quad \text{at} \quad \begin{aligned} &x=0, L_x \\ &y=0, L_y \\ &z=0, L_z \end{aligned} \quad , \tag{3.2}$$

where $\hat{n}_0$ is the local normal to the ideal surface. The solution for p_0 in terms of Mach number is

$$\frac{p_0}{\rho_0 c_0^2} = M \cos k_x x \cos k_y y \cos k_z z \cos(\omega_q t + \phi) \quad , \tag{3.3}$$

where

$$k_x = \frac{n\pi}{L_x} \tag{3.4}$$

$$k_y = \frac{m\pi}{L_y}$$

$$k_z = \frac{\ell\pi}{L_z}$$

and

$$\left(\frac{\omega_q}{c_0}\right)^2 = k^2 = \sqrt{k_x^{\,2} + k_y^{\,2} + k_z^{\,2}} \quad , \tag{3.5}$$

where ω_q is the frequency associated with the $(n,m,\ell;q)$ mode.

C. CLASSICAL SOLUTION FOR REAL BOUNDARIES

Since the cavity is assumed to have one perturbed wall, it is reasonable to expect the solution for real boundaries to differ from that for ideal boundaries. The classical linear solution for real boundaries, P_0 , may then be considered a summation of the classical linear solution for ideal boundaries plus perturbation correction terms due to the irregular boundary:

$$P_0 = p_0 + \delta p_1 + \delta^2 p_2 + \ldots \quad .$$

As long as the magnitude of the boundary perturbation is kept small, terms higher than first-order can be considered insignificant, so that

$$P_0 \doteq p_0 + \delta p_1 \quad \text{(to first order)} \tag{3.6}$$

and must be a solution to

$$\Box^2 P_0 = 0 \tag{3.7}$$

with boundary conditions

$$\nabla P_0 \cdot \hat{n} \;=\; 0 \qquad \text{at} \quad x=0,\; L_x[1+\delta f(y)] \qquad (3.8)$$
$$y=0,\; L_y$$
$$z=0,\; L_z \quad ,$$

where $\hat{n}$, the local normal to the real surface, is obtained by taking the divergence of the equation for the perturbed boundary, i.e., $\nabla \cdot \{x - L_x[1+\delta f(y)]\}$. Thus, to first order in δ,

$$\hat{n} \doteq \hat{x} - L_x \delta \frac{\partial f(y)}{\partial y} \hat{y} \quad ,$$

and when the above is used in (3.8) the result is

$$\frac{\partial P_0}{\partial x}\bigg|_{x=L_x[1+\delta f(y)]} - L_x \delta \frac{\partial P_0}{\partial y} \frac{\partial f(y)}{\partial y}\bigg|_{x=L_x[1+\delta f(y)]} = 0 \quad . \qquad (3.9)$$

D. THE APPLICABLE NON-LINEAR EQUATION

A real wave in a real cavity will of course have non-linearities introduced as it propagates through the cavity. Let P_1 be the correction term to the pressure due to the non-linearities of the medium. The total pressure P is then

$$P \;=\; P_0 + P_1 \qquad (3.10)$$

When (3.10) is inserted in (2.1), the LHS of the non-linear equation becomes

$$c_0^2 \Box_L^2 \frac{P_0}{\rho_0 c_0^2} + c_0^2 \Box_L^2 \frac{P_1}{\rho_0 c_0^2} \quad .$$

The first term above is known to be zero from (3.7), so the LHS of (2.1) becomes simply

$$c_0^2 \Box_L^2 \frac{P_1}{\rho_0 c_0^2} \quad .$$

Now if standing waves are to be produced in the cavity, P_0 must consist of terms of the form $\cos k_x x \cos k_y y \cos k_z z \cos \omega t$. Using the basic definitions

$$P_0 = -\rho_0 \frac{\partial \Phi_0}{\partial t}$$

and

$$\vec{U}_0 = \nabla \Phi_0$$

requires that $\vec{U}_0$ be of the form $\sin k_x x \sin k_y y \sin k_z z \sin \omega t$. When this information is used, the RHS of (2.1) will consist of four terms:

1. $-\dfrac{\partial^2}{\partial t^2} (\cos k_x x \cos k_y y \cos k_z z \cos \omega t)^2$

2. $- \frac{\partial^2}{\partial t^2} (\sin k_x x \sin k_y y \sin k_z z \sin \omega t)^2$

3. $\nabla^2 (\cos k_x x \cos k_y y \cos k_z z \cos \omega t)^2$

4. $\nabla^2 (\sin k_x x \sin k_y y \sin k_z z \sin \omega t)^2$

Although the expansion of these will produce many terms, it must be noted that only some will produce resonant standing waves. All the other terms may therefore be considered to give a small contribution when compared to the resonant terms, and thus may be neglected. This fact, when used with the trigonometric identities

$$\cos^2 \alpha = \frac{1}{2} [1 + \cos 2\alpha]$$

and

$$\sin^2 \alpha = \frac{1}{2} [1 - \cos 2\alpha]$$

reduces the RHS of (2.1) to a single term, and after a good deal of manipulation the non-linear equation may be approximated by

$$c_0^2 \Box_L^2 \frac{P_1}{\rho_0 c_0^2} \doteq - \frac{\gamma + 1}{2} \frac{\partial^2}{\partial t^2} \left(\frac{P_0}{\rho_0 c_0^2}\right)^2 \quad . \qquad (3.11)$$

The governing non-linear equation is now in a relatively simple form, and Appendix I develops the general form of the solution for P_1 .

IV. THE PERTURBED BOUNDARY

A. DEVELOPMENT OF A BOUNDARY CONDITION ON p_1

Recall that p_1 is the first-order correction term due to one boundary being irregular. Although it is known that for a perfectly rigid-walled cavity the pressure must be a maximum at the walls, this fact may not be conveniently used with one wall having an irregular form such as $L_x[1 + \delta f(y)]$. It would be highly desirable, therefore, to develop a boundary condition on p_1 to be applied at L_x . This may be accomplished as follows:

Using the incremental form for a Taylor series expansion on P_0 evaluated at the real boundary $L_x[1 + \delta f(y)]$ produces

$$P_0\Big|_{x=L_x[1+\delta f(y)]} =$$

$$P_0\Big|_{x=L_x} + \frac{\partial P_0}{\partial x}\Big|_{x=L_x} L_x\delta f(y) + \frac{1}{2}\frac{\partial^2 P_0}{\partial x^2}\Big|_{x=L_x} [L_x\delta f(y)]^2 + \ldots \quad .$$

Substitution of

$$P_0 \doteq p_0 + \delta p_1 \qquad (3.6)$$

on the RHS, and taking the partial derivative with respect to x on both sides gives

$$\left.\frac{\partial P_0}{\partial x}\right|_{x=L_x[1+\delta f(y)]} = \left.\frac{\partial p_0}{\partial x}\right|_{x=L_x} + \delta \left.\frac{\partial p_1}{\partial x}\right|_{x=L_x} + \left.\frac{\partial^2 p_0}{\partial x^2}\right|_{x=L_x} L_x \delta f(y) + \ldots \quad (4.1)$$

Following exactly the same procedure for the y coordinate gives

$$\left.\frac{\partial P_0}{\partial y}\right|_{y=L_x[1+\delta f(y)]} = \left.\frac{\partial p_0}{\partial y}\right|_{x=L_x} + \delta \left.\frac{\partial p_1}{\partial y}\right|_{x=L_x} + \left.\frac{\partial^2 p_0}{\partial y^2}\right|_{x=L_x} L_x \delta f(y) + \ldots \quad (4.2)$$

Recall that (3.9) required

$$\left.\frac{\partial P_0}{\partial x}\right|_{x=L_x[1+\delta f(y)]} - L_x \delta \left.\frac{\partial P_0}{\partial y} \frac{\partial f(y)}{\partial y}\right|_{x=L_x[1+\delta f(y)]} = 0 \quad .$$

When (4.1) and (4.2) are used in (3.9) the result is

$$\left.\frac{\partial p_0}{\partial x}\right|_{x=L_x} + \left.\frac{\delta \partial p_1}{\partial x}\right|_{x=L_x} + \left.\frac{\partial^2 p_0}{\partial x^2}\right|_{x=L_x} L_x \delta f(y) + \ldots$$

$$- L_x \delta \frac{\partial f(y)}{\partial y} [\left.\frac{\partial p_0}{\partial y}\right|_{x=L_x} + \left.\frac{\delta \partial p_1}{\partial y}\right|_{x=L_x} + \left.\frac{\partial^2 p_0}{\partial y^2}\right|_{x=L_x} L_x \delta f(y) + \ldots] = 0.$$

Collecting like-order terms in δ then gives

$$\left.\frac{\partial p_1}{\partial x}\right|_{x=L_x} + \left.\frac{\partial^2 p_0}{\partial x^2}\right|_{x=L_x} L_x f(y) - L_x \frac{\partial f(y)}{\partial y} \left.\frac{\partial p_0}{\partial y}\right|_{x=L_x} = 0,$$

or, as a boundary condition on p_1 applied at L_x,

$$\left.\frac{\partial p_1}{\partial x}\right|_{x=L_x} = L_x [-f(y) \left.\frac{\partial^2 p_0}{\partial x^2}\right|_{x=L_x} + \frac{\partial f(y)}{\partial y} \left.\frac{\partial p_0}{\partial y}\right|_{x=L_x}]. \qquad (4.3)$$

B. A FOURIER SERIES REPRESENTATION

It will be desirable for further development to represent the RHS of (4.3) as a Fourier series in cosines, so that p_1 may be expressed as a summation of normal modes. First, rewrite (4.3) as

$$\left.\frac{\partial}{\partial x}\left(\frac{p_1}{\rho_0 c_0^2}\right)\right|_{x=L_x} = M L_x [-f(y) \left.\frac{\partial^2 p_0}{\partial x^2}\right|_{x=L_x} + \frac{\partial f(y)}{\partial y} \left.\frac{\partial p_0}{\partial y}\right|_{x=L_x}]. \qquad (4.4)$$

Let the RHS of (4.4) equal $S\cos k_z z \cos \omega t$ and express it as an infinite Fourier series as follows:

$$S(y)\cos k_z z \cos \omega t = \left[\frac{A_0}{2} + \sum_{N=1}^{\infty} A_N \cos \frac{N\pi y}{L_y}\right] \cos k_z z \cos \omega t,$$

where the A_N are given by

$$A_N = \frac{2}{L_y} \int_0^{L_y} S(y) \cos \frac{N\pi y}{L_y}\, dy, \qquad N = 0,1,2,\ldots.$$

Then (4.4) becomes

$$\left.\frac{\partial}{\partial x}\left(\frac{p_1}{\rho_0 c_0^2}\right)\right|_{x=L_x} = \left(\frac{A_0}{2} + \sum_{N=1}^{\infty} A_N \cos \frac{N\pi y}{L_y}\right) \cos k_z z \cos \omega t.$$

The solution for p_1 is

$$\frac{p_1}{\rho_0 c_0^2} = - \sum_N \frac{\cos k_x' x}{k_x' \sin k_x' L_x} A_N \cos \frac{N\pi y}{L_y} \cos k_z z \cos \omega t \qquad (4.5)$$

where for $N = 0$ the coefficient is $A_0/2$, and k_x' is such that

$$k_x' = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{N\pi}{L_y}\right)^2 - k_z^2}\ .$$

Note that k_x' is not in the form of a standing wave in the cavity required by

$$k_x = \frac{n\pi}{L_x} . \qquad (3.4)$$

Thus the terms involving k_x' in (4.5) must also be expressed as a Fourier series:

$$\frac{\cos k_x'x}{k_x' \sin k_x' L_x} = \frac{B_0}{2} + \sum_{M=1}^{\infty} B_M \cos \frac{M\pi x}{L_x} , \qquad (4.6)$$

where the B_M are given by

$$B_M = \frac{2}{L_x} \int_0^{L_x} \frac{\cos k_x'x}{k_x' \sin k_x'L_x} \cos \frac{M\pi x}{L_x} dx, \quad M = 0,1,2,...$$

When the results of (4.6) are inserted into (4.5), p_1 is expressed as a double infinite series:

$$\frac{p_1}{\rho_0 c_0^2} = - \sum_N \sum_M B_M \cos \frac{M\pi x}{L_x} A_N \cos \frac{N\pi y}{L_y} \cos k_z z \cos \omega t, \quad (4.7)$$

where again each of the zero coefficients (A_0, B_0) are to be divided by a factor of two.

The formalism has now been developed whereby given any driving mode and boundary irregularity, p_1 - the first-order perturbation correction term - may be expressed in terms of the spatial components of the normal modes of the ideal cavity.

V. A SPECIFIC EXAMPLE

A. THE COSINUSOIDALLY PERTURBED BOUNDARY

As a specific example utilizing the theory developed in the preceeding sections, assume the cavity is driven in the (0,1,0;1) mode. Thus

$$\frac{p_0}{\rho_0 c_0^2} = M \cos k_y y \cos \omega t \quad . \tag{5.1}$$

Further assume that the boundary is perturbed cosinusoidally such that

$$f(y) = \cos \frac{\pi}{L_y} y \quad . \tag{5.2}$$

The boundary condition on p_1 then becomes

$$\frac{\partial}{\partial x} \left(\frac{p_1}{\rho_0 c_0^2}\right)\Bigg|_{x=L_x} = M \frac{L_x}{L_y} \pi k_y \sin \frac{\pi y}{L_y} \sin k_y y \cos \omega t$$

which, when used with the requirement that $k_y = \frac{\pi}{L_y}$, becomes

$$\frac{\partial}{\partial x} \left(\frac{p_1}{\rho_0 c_0^2}\right)\Bigg|_{x=L_x} = M \frac{L_x}{L_y} \frac{\pi}{2} k_y \cos \omega t - M \frac{L_x}{L_y} \frac{\pi}{2} k_y \cos \frac{2\pi y}{L_y} \cos \omega t \ .$$

By inspection, the solution for p_1 must be

$$\frac{p_1}{\rho_0 c_0^{\,2}} = -\,M\,\frac{L_x}{L_y}\,\frac{\pi}{2}\,\frac{k_y}{k_x{}'}\,\frac{\cos k_x{}'x}{\sin k_x{}'L_x}\cos\omega t$$

$$+\,M\,\frac{L_x}{L_y}\,\frac{\pi}{2}\,\frac{k_y}{k_x{}''}\,\frac{\cos k_x{}''x}{\sin k_x{}''L_x}\cos\frac{2\pi y}{L_y}\cos\omega t \qquad (5.3)$$

where $k_x{}'$ satisfies the equation

$$k_x{}' = \frac{\omega}{c} \qquad (5.4)$$

and $k_x{}''$ satisfies

$$(k_x{}'')^2 + (k_y + \frac{\pi}{L_y})^2 = (\frac{\omega}{c})^2 \quad . \qquad (5.5)$$

Solution for p_1 allows the creation of $P_0 \doteq p_0 + \delta p_1$,

$$\frac{P_0}{\rho_0 c_0^{\,2}} \doteq M\cos k_y y\cos\omega t - \delta M\,\frac{L_x}{L_y}\,\frac{\pi}{2}\,\frac{k_y}{k_x{}'}\,\frac{\cos k_x{}'x}{\sin k_x{}'L_x}\cos\omega t$$

$$+\,\delta M\,\frac{L_x}{L_y}\,\frac{\pi}{2}\,\frac{k_y}{k_x{}''}\,\frac{\cos k_x{}''x}{\sin k_x{}''}\cos\frac{2\pi y}{L_y}\cos\omega t \quad . \qquad (5.6)$$

Let the cavity dimensions be designed such that the (0,2,0;2) and the (1,1,0;2) modes will be degenerate. This requires that

$$(2\frac{\pi}{L_y})^2 = (\frac{2\omega}{c})^2$$

and

$$(\frac{\pi}{L_x})^2 + (\frac{\pi}{L_y})^2 = (\frac{2\omega}{c})^2 \quad .$$

Solution of the above equations yields

$$\frac{L_y}{L_x} = \sqrt{3} \tag{5.7}$$

for the desired degenerate modes. When (5.7) is inserted into (5.6), the result is

$$\frac{P_0}{\rho_0 c_0^2} = M \cos k_y y \cos \omega t - \delta M \frac{1}{\sqrt{3}} \frac{\pi}{2} \frac{k_y}{k_x'} \frac{\cos k_x' x}{\sin k_x' L_x} \cos \omega t$$

$$+ \delta M \frac{1}{\sqrt{3}} \frac{\pi}{2} \frac{k_y}{k_x''} \frac{\cos k_x'' x}{\sin k_x'' L_x} \cos \frac{2\pi y}{L_y} \cos \omega t \quad . \tag{5.8}$$

This must be squared when substituted into (3.11). Recalling that

$$P_0 \doteq p_0 + \delta p_1 \quad , \tag{3.6}$$

then the non-linear term has the form

$$\left(\frac{p_0}{\rho_0 c_0^2}\right)^2 = \left(\frac{p_0}{\rho_0 c_0^2}\right)^2 + 2\delta\left(\frac{p_0}{\rho_0 c_0^2}\right)\left(\frac{p_1}{\rho_0 c_0^2}\right) + \delta^2\left(\frac{p_1}{\rho_0 c_0^2}\right). \qquad (5.9)$$

The third term on the RHS above is of order δ^2 and may therefore be neglected. The first term becomes

$$\left(\frac{p_0}{\rho_0 c_0^2}\right)^2 = M^2 \cos^2 k_y y \cos^2 \omega t.$$

Using the trigonometric identity

$$\cos^2 \alpha = \frac{1}{2}\,[1 + \cos 2\alpha]$$

produces the following for p_0^2:

$$\left(\frac{p_0}{\rho_0 c_0^2}\right)^2 = \frac{M^2}{4}\,[1 + \cos 2k_y y]\,[1 + \cos 2\omega t] .$$

Since the only significant contribution will come from terms near resonance, those terms not involving a product of $\cos 2k_y y \cos 2\omega t$ may be neglected. Thus

$$\left(\frac{p_0}{\rho_0 c_0^2}\right)^2 \doteq \frac{M^2}{4} \cos 2k_y y \cos 2\omega t . \qquad (5.10)$$

Similarly, keeping just those terms which will lead to resonance produces

$$2\delta \frac{p_0}{\rho_0 c_0^2} \frac{p_1}{\rho_0 c_0^2} = - \frac{\gamma+1}{2} \delta M^2 (2\omega)^2 \frac{1}{\sqrt{3}} \frac{\pi}{2} \frac{k_y}{k_x'} \frac{\cos k_x' x}{\sin k_x' L_x} \cos k_y y \cos 2\omega t$$

$$+ \frac{\gamma+1}{2} \delta M^2 (2\omega)^2 \frac{1}{\sqrt{3}} \frac{\pi}{2} \frac{k_y}{k_x''} \frac{\cos k_x'' x}{\sin k_x'' L_x} \cos k_y y \cos 2\omega t \ . \tag{5.11}$$

Inserting the results of (5.9 - 5.11) into (3.11) gives

$$c_0^2 \Box_L^2 \frac{p_1}{\rho_0 c_0^2} = \frac{\gamma+1}{2} M^2 \omega^2 \cos 2k_y y \cos 2\omega t$$

$$- \frac{\gamma+1}{2} \delta M^2 (2\omega)^2 \frac{1}{\sqrt{3}} \frac{\pi}{2} \frac{k_y}{k_x'} \frac{\cos k_x' x}{\sin k_x' L_x} \cos k_y y \cos 2\omega t$$

$$+ \frac{\gamma+1}{2} \delta M^2 (2\omega)^2 \frac{1}{\sqrt{3}} \frac{\pi}{2} \frac{k_y}{k_x''} \frac{\cos k_x'' x}{\sin k_x'' L_x} \cos k_y y \cos 2\omega t. \tag{5.12}$$

The terms $\cos k_x' x$ and $\cos k_x'' x$ must be expressed as a cosine series of the form given by (4.7). When this is done, and only those components of argument $\pi x/L_x$ are retained, then all terms driving the (1,1,0;2) mode will be isolated, as desired. The results are

$$k_x' = \pi/L_y \tag{5.13a}$$

$$\cos k_x{'}x = 0.535 \cos \frac{\pi x}{L_x} , \tag{5.13b}$$

$$\sin k_x{'}L_x = 0.971 , \tag{5.13c}$$

$$k_x{''} = \frac{i\sqrt{3}\,\pi}{L_y} \tag{5.13d}$$

$$\cos k_x{''}x = -3.677 \cos \frac{\pi x}{L_x} , \tag{5.13e}$$

and

$$\sin k_x{''}L_x = i23.1 \tag{5.13f}$$

where use was made of the identities

$$\sin iA = i \sinh A$$

and

$$\sinh A = \frac{1}{2} [e^A - e^{-A}].$$

When the results of (5.13) are inserted into (5.12), the result is

$$c_0{}^2 \Box_L^2 \frac{p_1}{\rho_0 c_0{}^2} = \frac{\gamma+1}{2} M^2 \omega^2 \cos 2k_y y \cos 2\omega t$$

$$- \delta 1.83 \frac{\gamma+1}{2} M^2 \omega^2 \cos \frac{\pi x}{L_x} \cos \frac{\pi y}{L_y} \cos 2\omega t, \tag{5.14}$$

where the first term on the RHS is called the finite amplitude term, and the second term on the RHS shows the effects of the boundary irregularity on the first order non-linear perturbation correction.

B. THE SOLUTION FOR P_1

Appendix I shows the general solution for a function operated on by $c^2 \Box_L^2$. Using those results, the solution for P_1 for this example becomes immediate: for the finite amplitude term, designated by subscript f,

$$\left(\frac{P_1}{\rho_0 c_0^2}\right)_f = \frac{\gamma+1}{2} \frac{Q_f \sin \beta_f}{(2\omega)^2} M^2 \omega^2 \cos 2k_y y \cos (2\omega t+\beta_f), \quad (5.15)$$

and for the boundary irregularity induced term, designated by subscript p,

$$\left(\frac{P_1}{\rho_0 c_0^2}\right)_p = -1.83\ \delta\ \frac{\gamma+1}{2} M^2 \omega^2 \frac{Q_p \sin \beta_p}{(2\omega)^2} \cos k_x x \cos k_y y \cos(2\omega t+\beta_p). \quad (5.16)$$

When (5.15) and (5.16) are combined, they form P_1, the first order non-linear perturbation correction to the pressure. P_{1_f} is a non-linearly generated term which results from the existance of a 010 standing wave in the cavity. P_{1_p} exists because the irregular boundary forces a non-linear interaction between the correction term p_1 and the 010 mode.

It should be noted that there exists an additional correction term to the total pressure. This is the classical

correction to the 020 standing wave which exists in a cavity with an irregular wall. This additional correction term will be on the same order of magnitude as P_{1_p} [7], and must be taken into account when finding the total effect of the irregular boundary on the pressure. The following discussion of results will consider only the correction term generated by this example.

C. RESULTS

In order to best illustrate the magnitude of the effect of the non-familial mode in this example, recall that from (5.15) and (5.16),

$$\frac{P_1}{\rho_0 c_0^2} = \frac{\gamma+1}{2} \frac{Q_f \sin \beta_f}{(2\omega)^2} M^2 \omega^2 \cos 2k_y y \cos (2\omega t+\beta_f)$$

$$- 1.83\delta \frac{\gamma+1}{2} \frac{Q_p \sin \beta_p}{(2\omega)^2} M^2 \omega^2 \cos k_x x \cos k_y y \cos (2\omega t+\beta_p) .$$

Dividing both sides of the equation by

$$\frac{\gamma+1}{2} \frac{M^2}{4}$$

allowing the quality factors Q_f and Q_p to be equal, and assuming measurements are made at one of the corners so that

$$\cos k_x x = \cos k_y y = \cos 2k_y y = 1,$$

gives

$$\frac{P_1/\rho_0 c_0^2}{\frac{\gamma+1}{2}\frac{M^2}{4} Q_f} = \sin \beta_f \cos (2\omega t+\beta_f) - 1.83\delta \sin \beta_p \cos (2\omega t+\beta_p).$$

Let the LHS be designated by C. Additionally, introduce

$$H = 2 \frac{Q_f Q_p}{Q_f + Q_p} \left(\frac{\omega_f - \omega_p}{\omega_p}\right)$$

as a measure of the overlap of the curves obtained by plotting the magnitude of C versus a function of frequency for two cases:

1 - $\delta = 0$, corresponding to no non-familial mode existance, and

2 - $\delta \neq 0$, corresponding to a degeneracy between the (0,2,0;2) and (1,1,0;2) modes.

These curves, Figs. 2-6, allow a comparison of the magnitude of the P_1 term with and without the presence of a non-familial mode. The function of frequency on the abscissa is

$$F = \cot \beta_f,$$

where β is as defined by (I.1). Plots are provided for five different values of H. Additionally, for the case H = -1

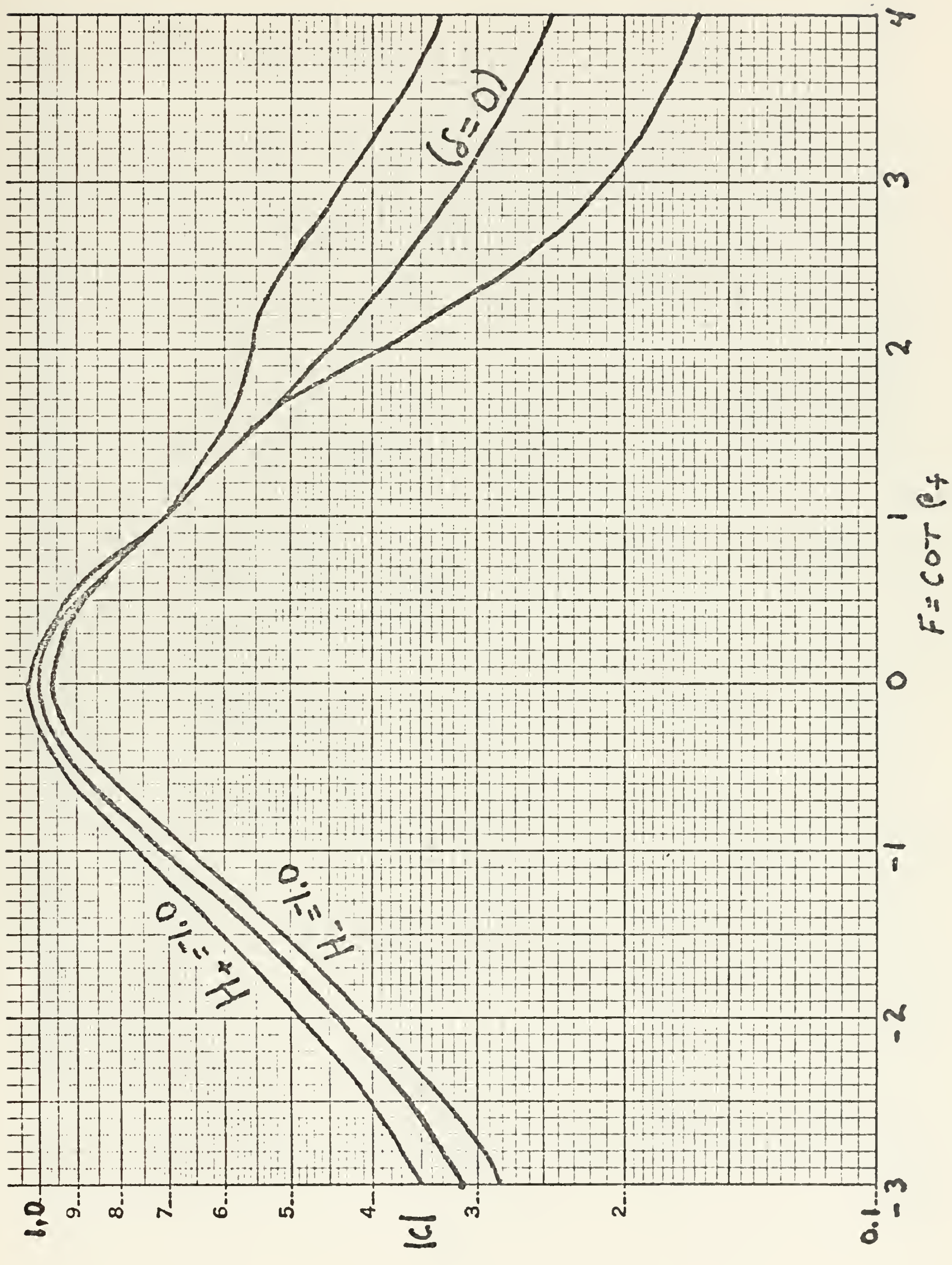

FIGURE 2

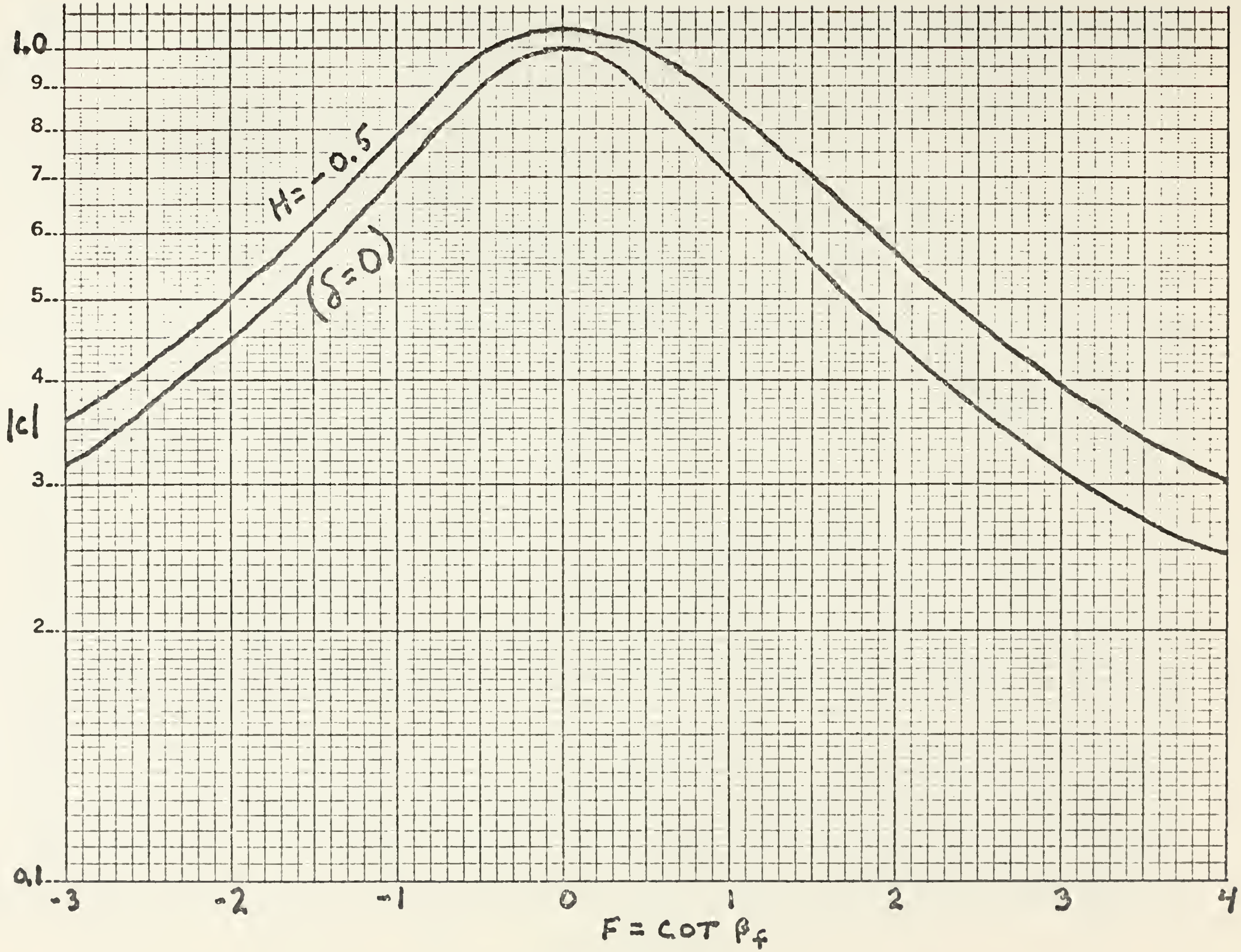

FIGURE 3

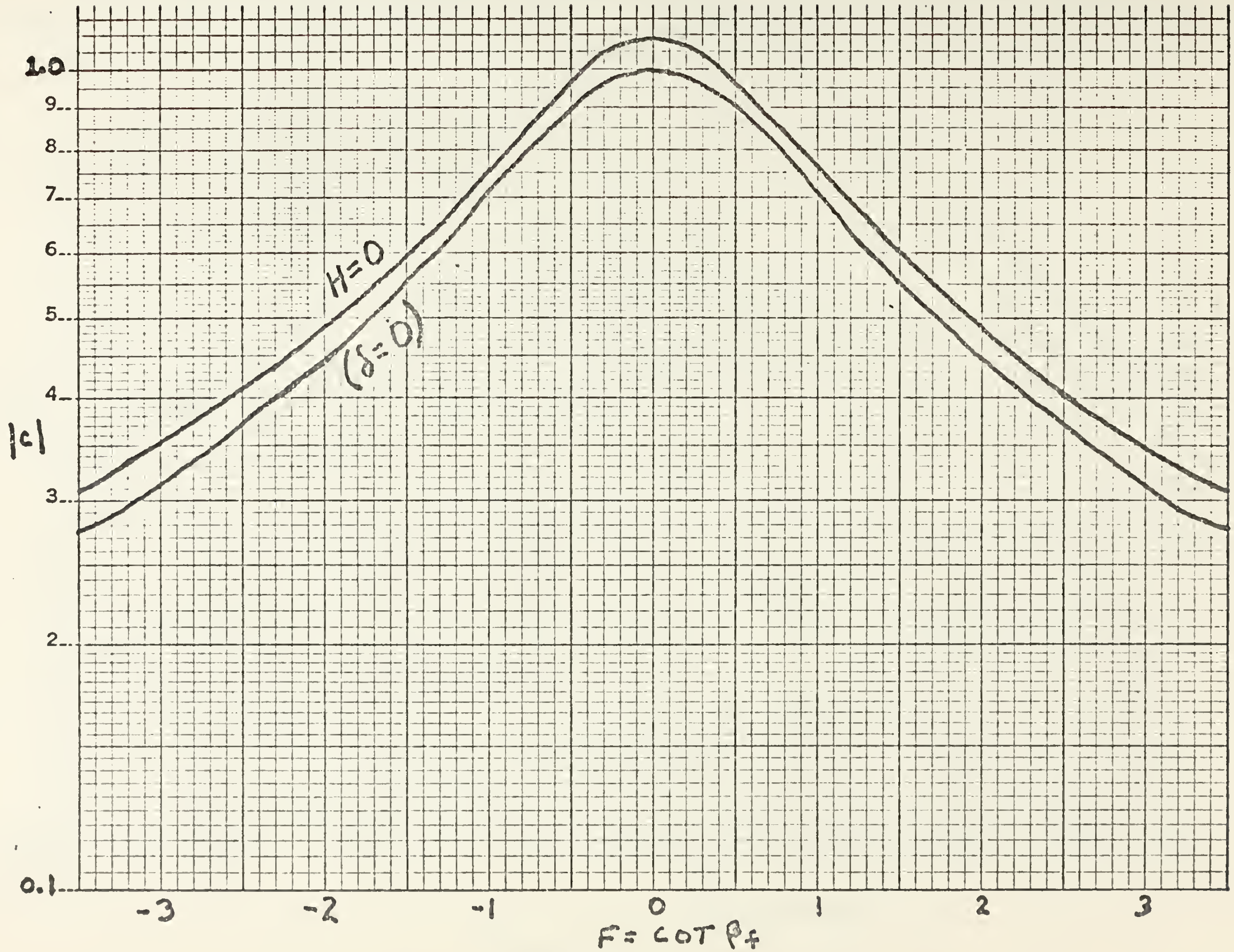

FIGURE 4

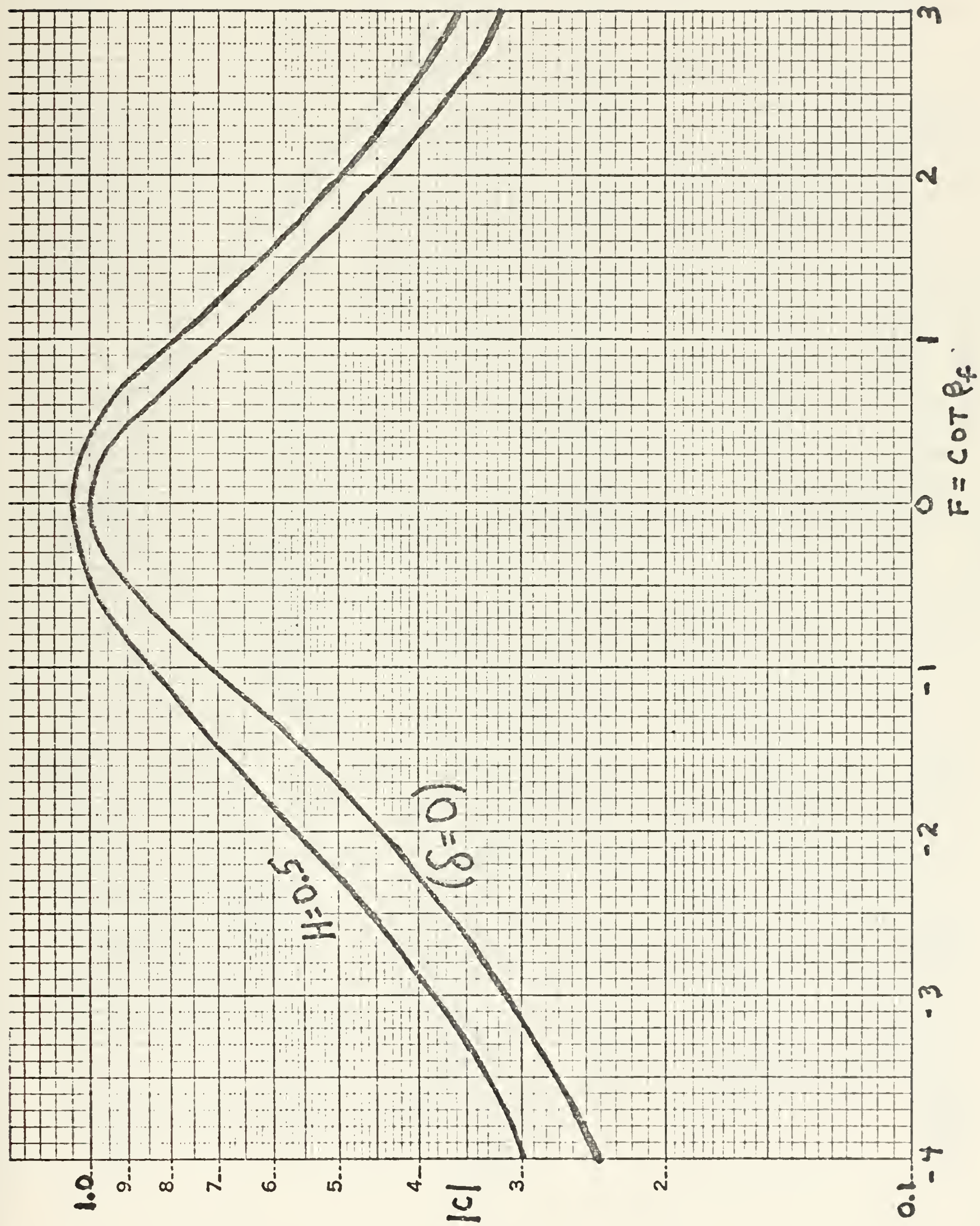

FIGURE 5

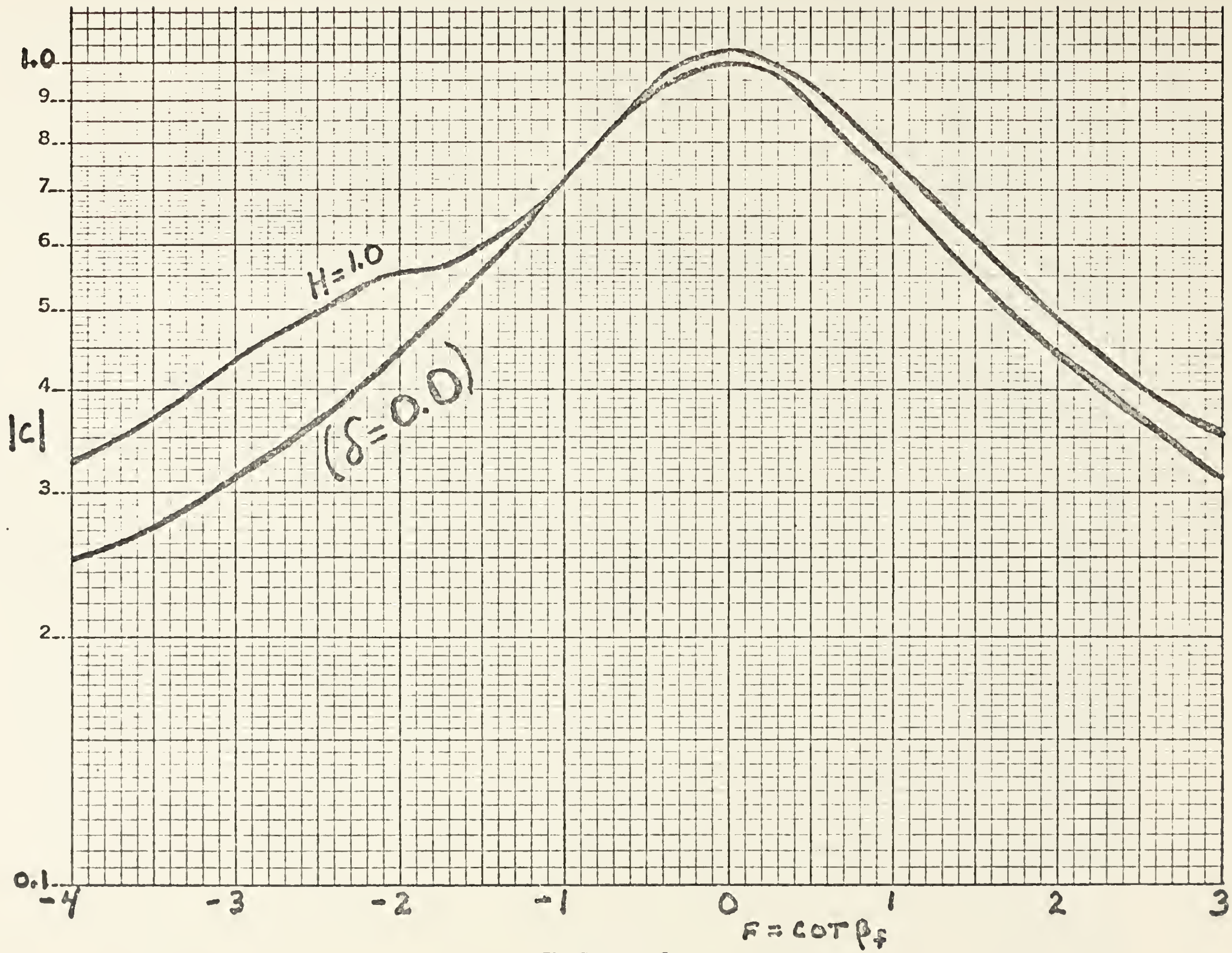

FIGURE 6

there is a plot labelled $H_- = -1$ which shows the result of selecting a different corner for measurements, one 180° out of phase with the plot labelled $H_+ = -1$. δ was arbitrarily chosen as 0.1.

From the graphs it is clear that the presence of a non-familial mode does affect the total solution for the pressure by an amount proportional to δ, the magnitude of the boundary perturbation.

VI. CONCLUSIONS

Non-linear theory has been applied to standing waves in a rigid-walled rectangular cavity with a perturbed boundary in an attempt to find one possible mechanism for the excitation of resonant modes other than those belonging to the family of the driven mode. It was observed that such an excitation does exist if the boundary perturbation and the dimensions of the cavity are favorably chosen. If the fractional perturbation of the boundary is given by δ, then the correction terms are of that same order.

APPENDIX I

It is desirable to show that when $c_p^2 \Box_L^2$ operates on a function of the form $\cos k_x x \cos k_y y \cos k_z z \cos(\omega t+\phi)$, the result will be of the form

$$\frac{\omega^2}{Q} \frac{1}{\sin \beta} \cos k_x x \cos k_y y \cos k_z z \cos(\omega t+\phi-\beta) ,$$

where β is an angle defined such that

$$\frac{1}{Q} / [1 - (\frac{c_p k}{\omega})^2] = \tan \beta . \qquad (I.1)$$

Begin by recalling that, by definition,

$$c_p^2 \Box_L^2 = c_p^2 \nabla^2 - \frac{\partial^2}{\partial t^2} - \frac{\omega}{Q} \frac{\partial}{\partial t} . \qquad (I.2)$$

Performing this operation on a function of the form $\cos k_x x \cos k_y y \cos k_z z \cos(\omega t+\phi)$ produces three terms, all with the common factors $\cos k_x x \cos k_y y \cos k_z z$. Canceling these like terms from both sides of (I.2), and using the fact that

$$k^2 = k_x^2 + k_y^2 + k_z^2$$

requires that

$$-c_p^{\,2}k^2 \cos(\omega t+\phi) + \omega^2\cos(\omega t+\phi) + \frac{\omega^2}{Q}\sin(\omega t+\phi) =$$

$$\frac{\omega^2}{Q}\frac{1}{\sin\beta}\cos(\omega t+\phi-\beta)$$

or, collection terms,

$$\omega^2[1 - \frac{c_p^{\,2}k^2}{\omega^2}]\cos(\omega t+\phi) + \frac{\omega^2}{Q}\sin(\omega t+\phi) = \frac{\omega^2}{Q}\frac{1}{\sin\beta}\cos(\omega t+\phi-\beta) \quad . \tag{I.3}$$

Rearrangement of (I.1) gives

$$[1 - \frac{c_p^{\,2}k^2}{\omega^2}] = \frac{\cos\beta}{Q\sin\beta} \quad ,$$

and substitution of this result into (I.3) requires

$$\frac{\omega^2}{Q}\frac{\cos\beta}{\sin\beta}\cos(\omega t+\phi) + \frac{\omega^2}{Q}\sin(\omega t+\phi) = \frac{\omega^2}{Q}\frac{1}{\sin\beta}\cos(\omega t+\phi-\beta) \quad .$$

Canceling the common factors ω^2/Q , and multiplying both sides by $\sin\beta$ then requires that

$$\cos\beta\cos(\omega t+\phi) + \sin\beta\sin(\omega t+\phi) = \cos(\omega t+\phi-\beta) \quad . \tag{I.4}$$

Using the trigonometric identity

$$\cos A \cos B + \sin A \sin B = \cos(A - B)$$

immediately shows (I.4) to be satisfied. Thus a general solution to $c_p^2 \Box_L^2$ when operating on a function of a specific form has been shown:

$$c_p^2 \Box_L^2 \cos k_x x \cos k_y y \cos k_z z \cos(\omega t+\phi) =$$

$$\frac{\omega^2}{Q} \frac{1}{\sin\beta} \cos k_x x \cos k_y y \cos k_z z \cos(\omega t+\phi-\beta)$$

where $\tan\beta = \frac{1}{Q} [1 - (\frac{c_p k}{\omega})^2]$.

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